

Monotone operator functions on C^* -algebra

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Abstract. The article is devoted to investigation of classes of functions monotone as functions on general C^* -algebras that are not necessarily the C^* -algebras of all bounded linear operators on a Hilbert space as it is in classical case of matrix and operator monotone functions. We show that for general C^* -algebras the classes of monotone functions coincide with the standard classes of matrix and operator monotone functions. For every class we give exact characterization of C^* -algebras that have this class of monotone functions, providing at the same time a monotonicity characterization of subhomogeneous C^* -algebras. We use this characterization to generalize one function based monotonicity conditions for commutativity of a C^* -algebra, to one function based monotonicity conditions for subhomogeneity. As a C^* -algebraic counterpart of standard matrix and operator monotone scaling, we investigate, by means of projective C^* -algebras and relation lifting, the existence of C^* -subalgebras of a given monotonicity class.

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1 Introduction.

The real-valued continuous function $f : I \mapsto \mathbb{R}$ on a (non trivial) interval $I \neq \mathbb{R}$ is called A -monotone for a given C^* -algebra A if for any $x, y \in A$ with spectrum in I ,

$$x \leq_A y \quad \Rightarrow \quad f(x) \leq f(y) \quad (1)$$

We denote by $P_A(I)$ the set of all A -monotone functions (defined on the interval I) for a C^* -algebra A . If $A = B(H)$, the standard C^* -algebra of all bounded linear operators on a Hilbert space H , then $P_A(I) = P_{B(H)}(I)$ is called the set of all operator monotone functions. If $A = M_n$, the standard C^* -algebra of all complex $n \times n$ matrices or equivalently of all (bounded) linear operators on an n -dimensional complex Hilbert space, then $P_n(I) = P_A(I) = P_{M_n}(I)$ is called the set of all matrix monotone functions of order n on an interval I . The set $P_n(I)$ consists of continuous functions on I satisfying (1) for pairs (x, y) of self-adjoint $n \times n$ matrices with spectrum in I . For each positive integer n , the proper inclusion $P_{n+1}(I) \subsetneq P_n(I)$ holds [1, 4]. For infinite-dimensional Hilbert space, the set of operator monotone functions on I can be shown to coincide with the intersection

$$P_\infty(I) = \bigcap_{n=1}^{\infty} P_n(I),$$

or in other words a function is operator monotone if and only if it is matrix monotone of order n for all positive integers n [5, Chap.5, Prop.5.1.5 (1)]. Keeping this in mind, for infinite-dimensional Hilbert space, we denote the class of operator monotone functions also by $P_\infty(I)$ or simply by P_∞ when the choice of the interval is clear from context. For the sake of clarity, if not stated otherwise we will assume that all C^* -algebras contain a unity. Formulations of most of the results can be adjusted to hold also in non-unital situation by the standard procedure of adjoining the unity, which amounts to adding a one-dimensional irreducible representation to the set of irreducible representations of A .

The Section 2 is devoted to description of the classes of monotone operator functions of C^* -algebras. We show that for general C^* -algebras the classes of monotone functions are the standard classes of matrix and operator monotone functions. For every such class we give exact characterization of C^* -algebras that have this class of monotone functions. This can be also used to give a monotonicity characterization of subhomogeneous C^* -algebras as discussed in [4, Theorem 5]. In Section 3 we use these characterizations to generalize one function based monotonicity condition for commutativity of a C^* -algebra, obtained by T. Ogasawara [11] and G. K. Pedersen [12], W. Wu [22], and Ji and Tomiyama [6], to one function based monotonicity condition for subhomogeneity. Finally in Section 4, we investigate, as a C^* -algebraic counterpart of standard matrix and operator monotone scaling, the existence of C^* -subalgebras of a given monotonicity class. We also state several problems motivated by the obtained results.

2 Scaling theorems

To begin with, note that for any C^* -algebra A there is a Hilbert space H such that $P_{B(H)} \subseteq P_A$, and in particular always $P_\infty \subseteq P_A$. Indeed, by Gelfand-Naimark construction

A is isometrically isomorphic to a C^* -subalgebra \tilde{A} of $B(H)$ for some Hilbert space H .

Any isomorphism between two C^* -algebras preserves the standard partial order induced by their positive cones. Therefore, any function which is operator monotone, that is $B(H)$ -monotone, is also \tilde{A} -monotone and hence A -monotone. In general, if $B \hookrightarrow A$ that is a C^* -algebra B is isomorphic to a C^* -subalgebra of a C^* -algebra A , then $P_A \subseteq P_B$. In other words the mapping $A \mapsto P_A$ is non-increasing. For the standard matrix imbedding scaling we have

$$M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \dots \hookrightarrow M_k \hookrightarrow \dots \hookrightarrow B(H).$$

This standard imbedding sequence is infinite and strictly increasing if $\dim H = \infty$, and we have the corresponding decreasing sequence

$$P_1(I) \supset P_2(I) \supset P_3(I) \supset \dots \supset P_n(I) \supset \dots \supset P_\infty(I).$$

The inclusions of function spaces $P_{n+1}(I) \subset P_n(I)$ and $P_\infty(I) \subset P_n(I)$ are strict for all positive integers n and non-trivial intervals I . Even though this fact has been known almost from the beginning of the theory of operator monotone functions, only recently explicit examples of functions from $P_n \setminus P_{n+1}$ for arbitrary choice of n have been constructed [4]. For general C^* -algebras, the imbedding partial order is more flexible allowing for different kinds of scalings.

The irreducible representations contain an important information about C^* -algebras, and dimensions of representations are the important classifying parameter. The following Lemma, which sharpens the assertion of [4, Theorem 5,(1) and (2)], is a key to further understanding of relationship between dimensions of irreducible representations and A -monotonicity for a C^* -algebra A on one side, and the operator monotonicity and matrix monotonicity on the other.

In the sequel, without loss of generality [4], we assume that $I = [0, \infty[$ and drop the interval from the corresponding notations.

Lemma 1. *Let A be a (unital) C^* -algebra.*

- 1) *If A has an irreducible representation of dimension n then any A -monotone function becomes n -matrix monotone, that is $P_A \subseteq P_n$.*
- 2) *If $\dim \pi \leq n$ for any irreducible representation π of A , then $P_n \subseteq P_A$.*
- 3) *If the set of dimensions of finite-dimensional irreducible representations of A is unbounded, then every A -monotone function is operator monotone, that is $P_A = P_\infty$.*
- 4) *If A has an infinite-dimensional irreducible representation, then every A -monotone function is operator monotone, that is $P_A = P_\infty$.*

Proof. Let $\pi : A \rightarrow M_n$ be an n -dimensional irreducible representation of A . Then irreducibility implies that $\pi(A) = M_n$. Thus for any pair $c, d \in M_n$, such that $0 \leq c \leq d$ there exists $a, b \in A$ such that $0 \leq a \leq b$ and $\pi(a) = c$ and $\pi(b) = d$. Then $f(a) \leq f(b)$ and hence $\pi(f(a)) \leq \pi(f(b))$ for any $f \in P_A(I)$. By continuity, $\pi(f(x)) = f(\pi(x))$ for

any $x \in A$. Thus $f(c) = f(\pi(a)) \leq f(\pi(b)) = f(d)$, and therefore $f \in P_n$. Hence, we have proved that $P_A \subseteq P_n$.

2) For any $f \in P_n$, for any $0 \leq a \leq b$ in A and for any irreducible representation $\pi : A \rightarrow M_m$, where $m \leq n$, we have $\pi(a) \leq \pi(b)$ in M_m . Then $\pi(f(a)) = f(\pi(a)) \leq f(\pi(b)) = \pi(f(b))$. If $0 \leq \pi(f(b) - f(a))$ for any irreducible representation π , then $\text{spec}(f(b) - f(a)) \in [0, \infty[$ that is $0 \leq f(b) - f(a)$ or equivalently $f(a) \leq f(b)$. Thus, $f \in P_A$ and we proved that $P_n \subseteq P_A$.

3) Let $\{\pi_j \mid j \in \mathbb{N} \setminus \{0\}\}$ be a sequence of irreducible finite-dimensional representations of A such that $n_j = \dim \pi_j \rightarrow \infty$ when $j \rightarrow \infty$. By 1) we have inclusion $P_A \subseteq P_{n_k}$ for any $k \in \mathbb{N} \setminus \{0\}$. Hence

$$P_A \subseteq \bigcap_{k \in \mathbb{N} \setminus \{0\}} P_{n_k} = \bigcap_{k \in \mathbb{N} \setminus \{0\}} P_k = P_\infty,$$

and since always $P_\infty \subseteq P_A$ holds, we get the equality $P_A = P_\infty$.

4) Let $\pi : A \rightarrow B(H)$ be irreducible representation of A on an infinite-dimensional Hilbert space H . By Kadison transitivity theorem, in the form it is stated in Takesaki's book [17, Ch.2, Theorem 4.18], $\pi(A)p = B(H)p$ for every projection $p : H \rightarrow H$ of a finite rank $n = \dim pH < \infty$. Let $B = \{a \in A \mid \pi(a)pH \subseteq pH, \pi(a)^*pH \subseteq pH\}$ be the C^* -subalgebra of A consisting of elements mapped by π to operators that, together with their adjoints, leave pH invariant. The restriction of $\pi : B \mapsto pB(H)p$ to B is n -dimensional representation of B on pH , and moreover it is irreducible and surjection, since $\pi(B)p = p\pi(B)p = p\pi(A)p = pB(H)p = B(pH)$. Thus 1) yields $P_A \subseteq P_B \subseteq P_n$, since B is a C^* -subalgebra of A . As the positive integer n can be chosen arbitrary, we get the inclusion

$$P_A \subseteq \bigcap_{n \in \mathbb{N} \setminus \{0\}} P_n = P_\infty.$$

Combining it with $P_\infty \subseteq P_A$ yields the equality $P_A = P_\infty$. ■

Corollary 1. *If $n_0 = \sup\{k \mid P_A \subseteq P_k\}$, then*

$$n_0 = n_1 = \sup\{\dim(\pi) \mid \pi \text{ is irreducible representation of } A\}.$$

Proof. By Lemma 1, the positive integer $n_0 = \sup\{k \mid P_A \subseteq P_k\}$ exists only if the set of dimensions of irreducible representations of A is bounded. Let

$$n_1 = \sup\{\dim(\pi) \mid \pi \text{ is irreducible representation of } A\}.$$

Then by 1) and 2) of Lemma 1 we have $P_{n_1} \subseteq P_A \subseteq P_{n_1}$, and hence $P_A = P_{n_1}$. Thus $P_{n_1} = P_A \subseteq P_{n_0}$ by 1) of Lemma 1. So, $n_1 \geq n_0$, and since $n_0 = \sup\{n \mid P_A \subseteq P_n\}$ and $P_A = P_{n_1}$, we get the desired $n_0 = n_1$. If $n_0 = \infty$, then $P_A = P_\infty$. By Lemma 1 either A has an infinite-dimensional irreducible representation or the set of dimensions of irreducible representations is unbounded, that is $n_0 = n_1 = \infty$, because if on the contrary the set of dimensions of irreducible representations is bounded by some positive integer n , then $P_n \subseteq P_A = P_\infty$, which is impossible since $P_\infty \subset P_n$ with $\text{gap } P_n \setminus P_\infty \neq \emptyset$. ■

Recall that a C^* -algebra A is said to be subhomogeneous if the set of dimensions of its irreducible representations is bounded. We say that A is n -subhomogeneous or subhomogeneous of degree n if n is the highest dimension of those irreducible representations of A .

Theorem 2. *Let A be a C^* -algebra. Then*

- 1) $P_A = P_\infty$ if and only if either the set of dimensions of finite-dimensional irreducible representations of A is unbounded, or A has an infinite-dimensional irreducible representation.
- 2) $P_A = P_n$ for some positive integer n if and only if A is n -subhomogeneous.

Proof. By Lemma 1 the only part of 1) left to prove is that $P_A = P_\infty$ implies that either the set of dimensions of finite-dimensional irreducible representations of A is unbounded, or A has an infinite-dimensional irreducible representation. Suppose on the contrary that

$$n_1 = \sup\{\dim(\pi) \mid \pi \text{ is irreducible representation of } A\} < \infty.$$

Then $P_A \subseteq P_{n_1}$ by Corollary 1, and $P_{n_1} \subseteq P_A$ by 2) of Lemma 1. Thus $P_A = P_{n_1}$. But there is a gap between P_∞ and P_n for any n . Hence $P_A \neq P_\infty$, in contradiction to the initial assumption $P_A = P_\infty$.

In part 2), again thanks to Lemma 1, it is left to prove that if $P_A = P_n$, then A is n -subhomogeneous. If $P_A = P_n$, then

$$n = n_0 = \sup\{k \in \mathbb{N} \mid P_A \subseteq P_k\}.$$

Indeed, if $n_0 > n$, then $P_{n_0} \subsetneq P_A = P_n$ since there exists a gap $P_m \subsetneq P_n$ for all $m > n$ as proved in [4]. But this contradicts to $P_A \subseteq P_{n_0}$ true by definition of n_0 . Hence $n_0 \leq n$. By Corollary 1,

$$n_0 = n_1 = \sup\{\dim(\pi) \mid \pi \text{ is irreducible representation of } A\},$$

and thus $n \leq n_0$. Therefor, $n = n_0 = n_1$ and so A is n -subhomogeneous. ■

Remark 1. A useful observation is that by Lemma 1 and Theorem 2, for any C^* -algebra A and any positive integer k , only two possibilities are possible, either $P_A \cap P_k = P_k$ or $P_A \cap P_k = P_A$.

The Theorem 2 can be used to extend 2) of Lemma 1 to be the "if and only if" statement also proved in [4].

Corollary 2. *Every matrix monotone function of order n is A -monotone if and only if the dimension of every irreducible representation of A is less or equal to n .*

Proof.

The "if" part is 2) of Lemma 1. To prove the "only if" part note that $P_n \subseteq P_A$ implies that $P_A = P_m$ for some $m \leq n$, and by 2) of Theorem 2, $m = n_1$. Thus $\dim(\pi) \leq n_1 = m \leq n$ for any irreducible representation of A . ■

If A is a commutative C^* -algebra, then every irreducible representation of A is one-dimensional and hence $P_A = P_1$, the set of all non-decreasing continuous functions. A natural class generalizing commutative C^* -algebras consists of n -homogeneous C^* -algebras, that is C^* -algebras with all non-zero irreducible representations being n -dimensional. The description of the class of monotone operator functions for n -homogeneous C^* -algebras follows from Theorem 2.

Corollary 3. *If C^* -algebra A is n -homogeneous, then $P_A = P_n$.*

Example 1. For any Hilbert space H , the equality $P_{B(H)} = P_{\dim H}$ holds. If $\dim H = n < \infty$, then $B(H) = M_n$ and $P_{B(H)} = P_n$; and if $\dim H = \infty$, then $P_{B(H)} = P_\infty$.

Example 2. The irrational rotation C^* -algebra A_θ is the C^* -algebra generated by two unitaries u and v satisfying commutation relation $uv = e^{i2\pi\theta}vu$ with some irrational $\theta \in]0, 1[$. It is isomorphic to the crossed product C^* -algebra $C(\mathbb{T}) \rtimes_{\sigma_\theta} \mathbb{Z}$ associated to the dynamical system consisting of the rotation σ_θ of the one-dimensional torus (the unite circle) \mathbb{T} by an angle $2\pi\theta$ with irrational θ . All non-zero irreducible representations of A_θ are infinite-dimensional since all points of \mathbb{T} are aperiodic under action of σ_θ . Hence $P_{A_\theta} = P_\infty$ by Theorem 2.

The rational rotation C^* -algebra A_θ is the crossed product C^* -algebra $C(\mathbb{T}) \rtimes_{\sigma_\theta} \mathbb{Z}$ where σ_θ is the rotation of \mathbb{T} by the angle $2\pi\theta$ with rational $\theta = \frac{m}{n}$ (m and n are relatively prime). The C^* -algebra A_θ is isomorphic to the C^* -algebra of cross-sections in the fibre bundle over \mathbb{T}^2 with fibre M_n , the $n \times n$ matrix algebra, and the structure group U_n , the n -dimensional unitary group. All points of \mathbb{T} are periodic of period n and thus all irreducible representations of A_θ are n -dimensional, which means that A_θ is an n -homogeneous C^* -algebra. Hence $P_{A_\theta} = P_n$ by Corollary 3.

Example 3. The C^* -algebras $C_0(X, M_n(\mathbb{C}))$ of continuous and vanishing at infinity $M_n(\mathbb{C})$ -valued functions on a locally compact Hausdorff space X are n -homogeneous C^* -algebras, and hence $P_{C_0(X, M_n(\mathbb{C}))} = P_n$ by Corollary 3. These C^* -algebras can be viewed as the space of continuous sections, vanishing at infinity, of the trivial M_n -bundle $X \times M_n$. In fact every n -homogeneous algebra arises as the algebra of continuous sections of some M_n -bundle [3, 18, 20].

Example 4. Let $A = C(\mathbb{T}) \rtimes_\sigma \mathbb{Z}$ be the crossed product algebra associated to the dynamical system consisting of a homeomorphism σ of \mathbb{T} . If σ is an orientation preserving homeomorphism of the circle without periodic points, then $n_1 = \infty$ and hence $P_A = P_\infty$ by Theorem 2.

Example 5. Let $A = C(X) \rtimes_\sigma \mathbb{Z}$ be the crossed product algebra associated to the dynamical system consisting of a homeomorphism σ of a compact Hausdorff space X . Then since any finite-dimensional irreducible representation of A is unitarily equivalent to an induced representation arising from a periodic point according to [19, Proposition 4.5] (see also [15]), the equality $P_A = P_\infty$ holds if and only if (X, σ) either have an aperiodic orbit or the set of periods of periodic points in (X, σ) is an unbounded subset of positive integers; and if all points of X are periodic for σ , and the set of periods is bounded, then $P_A = P_n$ for the maximal period n , coinciding with the maximal dimension for irreducible representations of A .

Example 6. Let \mathcal{H} be the three-dimensional discrete Heisenberg group represented by matrices,

$$\mathcal{H} = \left\{ \begin{pmatrix} 1, l, m \\ 0, 1, n \\ 0, 0, 1 \end{pmatrix} \mid l, m, n \in \mathbb{Z} \right\}.$$

Then it can be shown that the group C^* -algebra $C^*(\mathcal{H})$ is isomorphic to the crossed product C^* -algebra $C^*(\mathcal{H}) = C(\mathbb{T}^2) \rtimes_{\sigma} \mathbb{Z}$ associated to the homeomorphism of the two-dimensional torus \mathbb{T}^2 defined by $\sigma(s, t) = (s, t - s)$. This homeomorphism acts as rational rotation along the second coordinate direction if s is rational, and as irrational rotation if s is irrational. This means in particular that $C^*(\mathcal{H})$ has irreducible representations of infinite dimension and of any finite dimension. Hence $P_{C^*(\mathcal{H})} = P_{\infty}$ by Theorem 2.

Example 7. The c_0 -direct sum $A = \sum_{i=1}^{\infty} \oplus M_{n_i}$ of matrix algebras with a sequence of dimensions such that $n_i \rightarrow \infty$ when $i \rightarrow \infty$ is an example of a C^* -algebra for which there all irreducible representations are finite-dimensional, but the set of dimensions is unbounded. For this C^* -algebra $P_A = P_{\infty}$ by Theorem 2.

Example 8.

For any positive integer $n \geq 2$, the Cuntz C^* -algebra \mathcal{O}_n is the universal unital C^* -algebra on generators s_1, \dots, s_n satisfying relations

$$\begin{aligned} s_1 s_1^* + \dots + s_n s_n^* &= 1 \\ s_j^* s_k &= \delta_{jk} 1 = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases} \quad \text{for } j, k = 1, \dots, n. \end{aligned}$$

As \mathcal{O}_n has infinite-dimensional irreducible representations, $P_{\mathcal{O}_n} = P_{\infty}$ by 4) of Lemma 1. Actually \mathcal{O}_n is known to be an infinite-dimensional simple C^* -algebra.

3 Monotonicity characterizations of commutativity and sub-homogeneity

In C^* -algebras order induced by positivity is closely connected to algebraic properties. As an outcome of this, one can prove several unexpected results characterizing such properties as commutativity and sub-homogeneity in terms of monotonicity properties of functions.

There are several characterizations for the commutativity of C^* -algebras. One type is the well-known Stinespring theorem, that is, a C^* -algebra A is commutative if and only if every positive linear map from A to another C^* -algebra B (or from B to A) becomes completely positive [16]. To be precise, A becomes commutative if and only if every positive linear maps to B becomes two-positive (and then automatically completely positive). This is the beginning of the long and fruitful developments of understanding the matricial order structure of operator algebras (see for example [2]).

One of the first results in the direction of operator algebraic monotonicity of functions, obtained in 1955 by T. Ogasawara [11], states that if $0 \leq x \leq y$ implies $x^2 \leq y^2$ for all x, y in a C^* -algebra A , then A is commutative. A proof of this result can be found

also in the G. K. Pedersen's book [12, Proposition 1.3.9], as the main part of the proof of the more general statement saying that if $0 \leq x \leq y$ implies $x^\beta \leq y^\beta$ for all x, y in a C^* -algebra A and for a positive number $\beta > 1$, then A is commutative. In the present terminology, this result says that if the function $f(t) = t^\beta$ is A -monotone on the interval $[0, \infty[$ for some $\beta > 1$ then the C^* -algebra A is commutative. In 1998, W. Wu proved that if $f(t) = e^t$ is A -monotone, then the C^* -algebra A is commutative [22], by reducing the proof via involved approximation arguments to the A -monotonicity of the function t^2 and then using the Ogasawara's result [11]. In the recent paper by G. Ji and J. Tomiyama [6] it has been proved that the C^* -algebra is commutative if and only if all monotone functions are A -monotone, that is $P_1 = P_A$, and also if and only if there exists a continuous monotone function on the positive axis which is not matrix monotone of order 2 but A -monotone. If one makes the use of the operator monotonicity of the $\log t$ function noted already by C. Löwner [9], one can deduce from the A -monotonicity assumption for e^t , the A -monotonicity of the function t^β for any $\beta > 1$. Hence by the above cited result [12, Proposition 1.3.9] the C^* -algebra A has to be commutative which provides a short proof for the above mentioned result of Wu [22].

A C^* -algebra is commutative if and only if all its irreducible representations are one-dimensional, or in other words if and only if it is 1-homogeneous. In this sense both the n -homogeneous C^* -algebras and the n -subhomogeneous C^* -algebras, that is those C^* -algebras having only n -dimensional irreducible representations or respectively only irreducible representations of dimension less or equal to a positive integer n , are natural generalizations of the class of commutative C^* -algebras.

Using our results on relationship between homogeneity of C^* -algebras and the standard matrix monotonicity scaling of functions, we obtain an extension of the result of G. Ji and J. Tomiyama [6] to the n -subhomogeneous C^* -algebras.

Theorem 3. *Let A be a C^* -algebra. If there exists a pair of positive integers (m, n) obeying $n < m$, and such that firstly, every A -monotone function is n -monotone, that is $P_A \subset P_n$, and secondly, there is a function which is at the same time A -monotone, n -monotone but not m -monotone, that is $P_A \cap (P_n \setminus P_m) \neq \emptyset$, then there exists some intermediate integer $n \leq j < m$ such that*

- 1) *every A monotone function is j -monotone, that is $P_A = P_j$;*
- 2) *the C^* -algebra A is j -subhomogeneous.*

If $m = n + 1$, then we get the following useful specialization of Theorem 3.

Theorem 4. *Let A be a C^* -algebra. If there exists a positive integer n , such that firstly, every A -monotone function is n -monotone, that is $P_A \subset P_n$, and secondly, there is a function which is at the same time A -monotone, n -monotone but not $(n + 1)$ -monotone, that is $P_A \cap (P_n \setminus P_{n+1}) \neq \emptyset$, then $P_A = P_n$ and the C^* -algebra A is n -subhomogeneous.*

Proof. (Theorem 3 and 9) By Lemma 1, for any C^* -algebra A and any positive integer k , only two possibilities are possible, either $P_A \cap P_k = P_k$ or $P_A \cap P_k = P_A$. If there exists a positive integer n , such that $P_A \subset P_n$, then A is n_0 -subhomogeneous and $P_A = P_{n_0}$ by Corollary 1. We have that $P_{n_0} \cap (P_n \setminus P_m) = P_A \cap (P_n \setminus P_m) \neq \emptyset$ and $P_{n_0} = P_A \subseteq P_n$. Since $P_m \subsetneq P_n$ for all $m > n$, we get $P_m \subseteq P_{n_0} \subseteq P_n$ and $n \leq n_0 < m$ by assumption.

Theorem is obtained in the special case when $m = n + 1$. Indeed, in this case $P_{n_0} = P_A \subseteq P_n$ and $n \leq n_0 < n + 1$. Hence $n = n_0$ and $P_A = P_n$. Thus A becomes n -subhomogeneous. ■

Example 9. As we have mentioned before $f(t) = t^\beta \in P_1 \setminus P_2$ for $\beta > 1$. Hence if $f(t) = t^\beta \in P_A$ for some $\beta > 1$, then $P_A \cap (P_1 \setminus P_2) \neq \emptyset$, and by Theorem we get that $P_A = P_1$. Hence A is 1-homogeneous by Theorem 2, that is all its irreducible representations are one-dimensional. Indeed, this implies that A is commutative. This is the essential point of the arguments in Ji and Tomiyama [6], which yields the results of G. K. Pedersen [12, Proposition 1.3.9], T. Ogasawara [11] and W. Wu [22].

A complementing assertions to Theorem 3 is as follows. The corresponding specialization of Theorem 5 for $m = n + 1$ is obtained just by replacing m by $n + 1$.

Theorem 5. *Let A be a C^* -algebra. If for all pairs of positive integers (m, n) obeying $n < m$ there is no functions that are at the same time A -monotone, n -monotone but not m -monotone, that is $P_A \cap (P_n \setminus P_m) = \emptyset$, then*

- 1) *every A monotone function is operator monotone, that is $P_A = P_\infty$;*
- 2) *either the set of dimensions of finite-dimensional irreducible representations of A is unbounded, or A has an infinite-dimensional irreducible representation.*

Proof. Suppose that $P_A \neq P_\infty$ in spite of $P_A \cap (P_n \setminus P_m) = \emptyset$. Then by Lemma 1 all irreducible representations of A are finite-dimensional and the set of their dimensions is bounded. By Theorem 2 there exists a positive integer $k \geq n$ such that $P_A = P_k$. Since the existence of gaps asserts that $P_k \subsetneq P_{k+1} \neq \emptyset$, we have $P_k \setminus P_{k+1} \neq \emptyset$ and hence $P_A \cap (P_k \setminus P_{k+1}) = P_k \cap (P_k \setminus P_{k+1}) = (P_k \setminus P_{k+1}) \neq \emptyset$ in contradiction with the condition of the theorem. ■

Remark 2. The gaps between classes of monotone matrix functions were addressed in [1], and more recently, in [4] and [10]. In [10] the "if and only if" extension was obtained of the result on fractional mapping between classes of matrix monotone functions from the paper by Wigner and von Neumann [21], and then it was shown that this extended result yields a proof of the implication that if $n \geq 2$, then $P_n = P_{n+1}$ implies $P_n = P_\infty$. This can be viewed as a different prove for specialization of Theorem 5 to the case when $A = B(H)$ and $m = n + 1$.

Remark 3. In [14], a new proof of Löwner's theorem on integral representation of operator monotone functions, different from the three proofs by Löwner, Bendat and Sherman, and Karanyi and Nagi, has been obtained by employing another classes of functions \mathcal{M}_n in between P_n and P_{n+1} . A real-valued functions h on $(0, \infty)$ is in \mathcal{M}_n if and only if, for $a_j \in \mathbb{R}$, $\lambda_j > 0$ and $j = 1, \dots, 2n$, the following implication holds:

$$\left(\sum_{j=1}^{2n} a_j \frac{t\lambda_j - 1}{t + \lambda_j} \geq 0 \text{ for } t > 0, \sum_{j=1}^{2n} a_j = 0 \right) \Rightarrow \left(\sum_{j=1}^{2n} a_j h(\lambda_j) \geq 0 \right).$$

As important part of the proof of Löwner's theorem, it was shown in [14] that $P_{n+1} \subseteq \mathcal{M}_n \subseteq P_n$ for any positive integer n . There an explicit example, showing that $P_2 \setminus \mathcal{M}_2 \neq \emptyset$, has been pointed out, thus particularly implying that $P_2 \setminus P_3 \neq \emptyset$. Proving that $P_n \setminus \mathcal{M}_n \neq \emptyset$ and $\mathcal{M}_n \setminus P_{n+1} \neq \emptyset$ for an arbitrary n is still an open problem. Motivated by our results, we feel that the related problem of finding a C^* -algebraic interpretation and perhaps a C^* -algebraic generalization of the spaces \mathcal{M}_n would be of interest.

Theorem 3 can be used to obtain the following unexpected operator monotonicity based characterizations of subhomogeneous C^* -algebras and of dimension for Hilbert spaces.

Let $g_n(t) = t + \frac{1}{3}t^3 + \dots + \frac{1}{2n-1}t^{2n-1}$, where n is some positive integer. In [4] it was proved that there exists

$\alpha_n > 0$ such that $g_n \in P_n([0, \alpha_n]) \setminus P_{n+1}([0, \alpha_n])$, and consequently $f_n = g_n \circ h_n \in P_n \setminus P_{n+1}$, where $h_n(t)$ is the Möbius transformation $h_n(t) = \frac{\alpha_n t}{1+t}$, operator monotone on $[0, \infty[$, with the inverse $h_n^{-1}(t) = \frac{t}{\alpha_n - t}$ operator monotone on $[0, \alpha_n[$.

Corollary 4. *If f_n is A -monotone function on $[0, \infty[$ for a C^* -algebra A , then A is a subhomogeneous C^* -algebra, such that dimensions of all its irreducible representations do not exceed n .*

Corollary 5. *If f_n is $B(H)$ -monotone for some positive integer n and a Hilbert space H , then $\dim H \leq n$.*

Proof. (Corollary 4 and 5) By Remark 1, $P_A \subseteq P_n$ or $P_n \subseteq P_A$. If $P_A \subseteq P_n$, then $P_A = P_n$ by Theorem 9. Hence A is subhomogeneous by Theorem 2(2). If $P_n \subseteq P_A$, then there exists $k \leq n$ such that $P_A = P_k$, and dimensions of irreducible representations do not exceed n by Theorem 2(2). In the special case when $A = B(H)$, this property yields $\dim H \leq n$. ■

4 Existence of subalgebras respecting scaling

In this section we obtain some results on C^* -subalgebras and monotonicity, that can be viewed as a C^* -algebraic counterpart of the standard scaling $M_k \hookrightarrow M_n \hookrightarrow B(H)$, $k < n < \dim H = \infty$.

In the the following theorem CM_m means the cone of M_m , that is, $C_0([0, 1]) \otimes M_m = C_0([0, 1], M_m)$. In the proof we will make use of some results on projective C^* -algebras and lifting of relations in C^* -algebras [8, 7].

Theorem 6. *Let A be a C^* -algebra.*

- 1) *If A is a C^* -algebra having an n -dimensional irreducible representation for some positive integer n , then for any positive integer $m \leq n$ there exists an m -homogeneous (presumably nonunital) C^* -subalgebra B .*
- 2) *If A has an infinite-dimensional irreducible representation π , then*

- 2a) For any positive integer m there exists a C^* -subalgebra B in A , such that B is m -homogeneous.
- 2b) The C^* -algebra has ∞ -homogeneous C^* -subalgebra B , that is a C^* -subalgebra whose all non-zero irreducible representations are infinite-dimensional, if and only if A is not residually finite-dimensional, that is

$$I = \bigcap_{\pi} \text{Ker}(\pi) \neq \{0\},$$

where the intersection is taken over all finite-dimensional irreducible representations.

- 3) If the set of dimensions of finite-dimensional irreducible representations of A is unbounded, then for any positive integer m there exists m -homogeneous C^* -subalgebra of A .

Proof. 1) Let $\pi : A \rightarrow B(H)$ be an n -dimensional irreducible representation of A . Then $\pi(A)$ is isomorphic to $n \times n$ matrix algebra M_n . Let $\{e_{i,j}\}$ be the standard matrix units for $\pi(A)$ obtained from the standard matrix units for $n \times n$ matrix algebra via this isomorphism. Now for any positive integer $m \leq n$ the elements $a_2 = e_{2,1}, \dots, a_m = e_{m,1}$ satisfy the relations of Theorem 10.2.1 in the Loring's book [8], namely,

$$(*) \begin{cases} \|a_j\| \leq 1 \\ a_j a_k = 0 & (j, k = 2, \dots, m) \\ a_j^* a_k = \delta_{j,k} a_2^* a_2 & (j, k = 2, \dots, m) \end{cases}$$

Hence by the above cited theorem these elements are lifted to A keeping those relations. Thus there are elements $\bar{a}_2, \dots, \bar{a}_m$ in A satisfying the same relations such that $\pi(\bar{a}_j) = a_j$ for $2 \leq j \leq m$.

Let $B = C^*(\bar{a}_2, \dots, \bar{a}_m)$ be the C^* -algebra of A generated by $\bar{a}_2, \dots, \bar{a}_m$.

By Proposition 3.3.1 in [8], the universal C^* -algebra on generators c_2, \dots, c_m satisfying the same relations is isomorphic to CM_m by the map $c_j \mapsto t \otimes e_{j,1}$. Therefore there exists a homomorphism from CM_m onto B , and since CM_m is m -homogeneous, its image B must be m -homogeneous.

2a) Let π be an infinite-dimensional irreducible representation of A on a Hilbert space H . Take an m -dimensional projection $p : H \mapsto H$. By Kadison transitivity theorem, in the form it is stated in Takesaki's book [17, Ch.2, Theorem 4.18], $\pi(A)p = B(H)p$. Then $pB(H)p = p\pi(A)p \cong M_m$. The restriction of $\pi : B \mapsto pB(H)p$ to the C^* -subalgebra $B = \{a \in A \mid \pi(a)pH \subseteq pH, \pi(a)^*pH \subseteq pH\}$ of A consisting of elements mapped by π to operators that together with their adjoints leave pH invariant, is n -dimensional representation of B on pH , and moreover it is irreducible and surjection, since $\pi(B)p = p\pi(B)p = p\pi(A)p = pB(H)p = B(pH)$. Repeating the lifting argument from 1) with $k = m$, we get the m -homogeneous C^* -subalgebra of B and thus of A .

2b)↑: Suppose A is not residually finite-dimensional. Then

$$I = \bigcap_{\substack{\pi \in \text{irred.rep.}(A) \\ \dim \pi < \infty}} \text{Ker}(\pi) \neq 0$$

is an ideal and thus is a C^* -subalgebra in A . Let π be a non-zero irreducible representation of I on a Hilbert space H .

Since I is an ideal in A there exists an irreducible representation $\tilde{\pi}$ of A on the same Hilbert space H extending π , that is coinciding with π on I . If $\dim \tilde{\pi} = \dim \pi = \dim H$ is finite, then $\pi(I) = \tilde{\pi}(I) = 0$ by definition of I , in contradiction with assumption that π is non-zero. Hence, $\dim \pi = \infty$, and since π has been chosen arbitrary, $B = I$ is an ∞ -homogeneous C^* -subalgebra of A .

2b) \Downarrow : Assume that A is residually finite-dimensional, that is

$$I = \bigcap_{\substack{\pi \in \text{irred. rep.}(A) \\ \dim \pi < \infty}} \text{Ker}(\pi) = 0.$$

Then any C^* -subalgebra B of A has a non-zero finite-dimensional irreducible representation. Indeed, A being residually finite-dimensional has sufficiently many finite-dimensional irreducible representations, that is for any non-zero C^* -subalgebra B there exists an irreducible finite-dimensional representation $\tilde{\pi}$ of A such that $\tilde{\pi}(B) \neq \{0\}$. Since $\tilde{\pi}(A) \cong M_{\dim \tilde{\pi}}$, it holds that $\tilde{\pi}(B) = M_{k_1} \oplus \cdots \oplus M_{k_l}$ is the direct sum of full matrix algebras.

Then cutting down onto one of the summands by a central projection p_j yields a non-zero finite-dimensional irreducible representation $\pi : B \rightarrow p_j \tilde{\pi}(B) p_j$ of B . Hence, there is no ∞ -homogeneous C^* -subalgebras in A , if A is residually finite-dimensional.

3) If the set of dimensions of finite-dimensional irreducible representations of A is unbounded, then for any positive integer m there exists an irreducible representation of dimension $n > m$. As in 1), the elements $a_2 = e_{21}, \dots, a_m = e_{m1}$ of M_n can be lifted to the elements of A , which generate an m -homogeneous C^* -subalgebra in A . ■

Combining Lemma 1 with Theorem 6 we obtain the following result.

Theorem 7. *Let A be a C^* -algebra.*

- 1) *If A is a C^* -algebra having n -dimensional irreducible representation for some positive integer n , then for any positive integer $m \leq n$ there exists a C^* -subalgebra B , such that $P_B = P_m$.*
- 2) *If A has an infinite-dimensional irreducible representation π , or the set of dimensions of finite-dimensional irreducible representations of A is unbounded, then for any positive integer m there exists C^* -subalgebra B of A such that $P_B = P_m$.*
- 3) *If A is not residually finite dimensional, then there exists a C^* -subalgebra B such that $P_B = P_\infty$.*

Example 10. Let $A = C(X) \rtimes_\sigma \mathbb{Z}$ be the transformation group (crossed product) C^* -algebra associated to a dynamical system $\Sigma = (X, \sigma)$ consisting of a homeomorphism σ on a compact Hausdorff metric space X , and let $Per(\Sigma)$ denote the set of all periodic points of Σ , and $\overline{Per(\Sigma)}$ be its closure in X . In [19, Theorem 4.6] it was shown that

A is residually finite-dimensional if and only if $\overline{Per(\Sigma)} = X$. Hence, by Theorem 6 and Theorem 7, if $\overline{Per(\Sigma)} \neq X$, then $A = C(X) \rtimes_{\sigma} \mathbb{Z}$ contains some C^* -subalgebra B such that all irreducible representations of B are infinite-dimensional and $P_B = P_{\infty}$. The group C^* -algebra $C^*(\mathcal{H}) = C(\mathbb{T}^2) \rtimes_{\sigma} \mathbb{Z}$ of the three-dimensional Heisenberg group mentioned in Example 6 has property $\overline{Per(\Sigma)} = X$, and is therefore a residually finite-dimensional C^* -algebra. By Theorem 6, any C^* -subalgebra of $C^*(\mathcal{H})$ possesses finite-dimensional irreducible representation. Moreover $C^*(\mathcal{H})$ has infinite-dimensional irreducible representations (for example those induced by aperiodic points), and hence by Theorem 6 and Theorem 7, for any positive integer m it has to have m -homogeneous C^* -subalgebra B such that $P_B = P_m$. It could be interesting to construct explicitly such subalgebras in $C^*(\mathcal{H})$.

Example 11. The C^* -algebra of compact operators $K(H)$ on an infinite-dimensional Hilbert space H is not residually finite-dimensional. All irreducible representations of $K(H)$ are infinite-dimensional except for the zero representation. Hence

$$I = \bigcap_{\substack{\in \text{irred.rep.}(A) \\ \dim \pi < \infty}} \text{Ker}(\pi) = K(H),$$

and $K(H)$ itself can be taken as an example of a ∞ -homogeneous C^* -subalgebra of $K(H)$ such that

$$P_{K(H)} = P_{\infty}.$$

Example 12. Let A be a simple C^* -algebra, that is a C^* -algebra with no non-zero closed ideals, and assume that A has infinite-dimensional irreducible representation, thus implying that $P_A = P_{\infty}$ by 4) of Lemma 1. In this case every non-zero irreducible representation of A is naturally infinite-dimensional. In particular A is not residually finite-dimensional, and so has at least one ∞ -homogeneous C^* -subalgebra B such that $P_B = P_{\infty}$, namely $B = A$. The C^* -algebra of compact operators $K(H)$ on an infinite-dimensional Hilbert space H , the Cuntz C^* -algebras \mathcal{O}_n and the irrational rotation C^* -algebra are simple C^* -algebras which have infinite-dimensional irreducible representations. The question arising from these observations is whether any infinite-dimensional simple C^* -algebra contains a proper (different from the whole algebra) ∞ -homogeneous C^* -subalgebra, and how to find and classify such C^* -subalgebras for the specific examples where such subalgebras exist.

The following question is suggested by Theorems 6, 7 and 3: if A is n -matrix monotone, that is $P_A = P_n$, and its C^* -subalgebra B is k -matrix monotone for some $k < n$, that is $P_B = P_k$, then is it true that for any l between n and k there exists a C^* -subalgebra C containing B for which C is l -monotone ?

This question is closely related to the following question concerning representations of C^* -algebras: For A having irreducible representation of finite dimension n , its subalgebra B , sub-homogeneous of degree $k < n$, and any integer l between k and n , can we find a sub-homogeneous subalgebra of degree l including B ?

As an easy counter example to both questions let $A = M_4$ and $B = M_2 \oplus M_2$ be the direct sum of two M_2 , which is isomorphic to a maximal C^* -subalgebra of M_4 obtained

by placing the direct summands as diagonal blocks. In this case $P_A = P_4$ and $P_B = P_2$, but there is no C^* -subalgebra C of A containing B with $P_C = P_3$. At the same time, if $B = M_2 \oplus M_1 \oplus M_1$ imbedded in $A = M_4$ as diagonal blocks with non-increasing order of dimensions, then $P_A = P_4$, $P_B = P_2$ and there exists a C^* -subalgebra C of A containing B with $P_C = P_3$. Namely, one can take $C = M_3$ imbedded in $A = M_4$ by placing the direct summand as the diagonal block containing the $M_2 \oplus M_1$ part of B , and putting M_1 in the remaining diagonal spot. Thus even in the case of the matrix algebra M_n we have to consider the location of its C^* -subalgebra B in A to be able to assert the existence of the intermediate C^* -subalgebra C . In other words one needs many experiments in concrete C^* -algebras to clarify how the gaps appear depending on the nature of the imbeddings. We feel that the monograph [13], on subalgebras of C^* -algebras and the limit algebras of inclusion sequences, discussing the importance of the nature of the inclusions, contains results which could be of interest in this respect.

As we discussed before for a C^* -subalgebra B of a C^* -algebra A , the equality $P_B = P_1$ holds if and only if B is commutative (abelian). Maximal abelian C^* -subalgebras are important for understanding representations and structure of a C^* -algebra. Closely related to the previous discussion is the following problem. Let B be a maximal abelian C^* -subalgebra of a C^* -algebra A . What are the "allowed" positive integers j for which there exists a C^* -subalgebra C of A containing B such that $P_C = P_j$?

Beyond the class of full matrix algebras, there are many important examples of infinite decreasing inclusion sequences of C^* -subalgebras $A_1 \hookleftarrow A_2 \hookleftarrow A_3 \hookleftarrow \dots$. If for some positive integer k there exists a positive integer n such that $P_{A_k} = P_n$, then for all positive integers $j \geq k$ there exists a positive integer $l_j \leq n$ such that $P_{A_j} = P_{l_j}$, and moreover $l_{j+1} \leq l_j$ for all $j \geq n$. So the inclusion non-increasing sequence of function spaces P_{A_k} stabilizes at some positive integer $s \leq n$, which means that $P_{A_j} = P_s$ for all $j \geq s$. The first question arising from these considerations is whether it is possible to have $P_{A_\infty} = P_t$ for some $t < s$, where $A_\infty = \bigcap_{r \in \mathbb{N} \setminus \{0\}} A_r$. The second question is concerned with P_∞ . Suppose that $P_{A_k} = P_\infty$ for all positive integers k . Is it possible to have a decreasing sequence satisfying $P_{A_\infty} = P_\infty$, and if it is possible what are then the properties of sequences of C^* -subalgebras and their imbeddings leading to this situation?

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